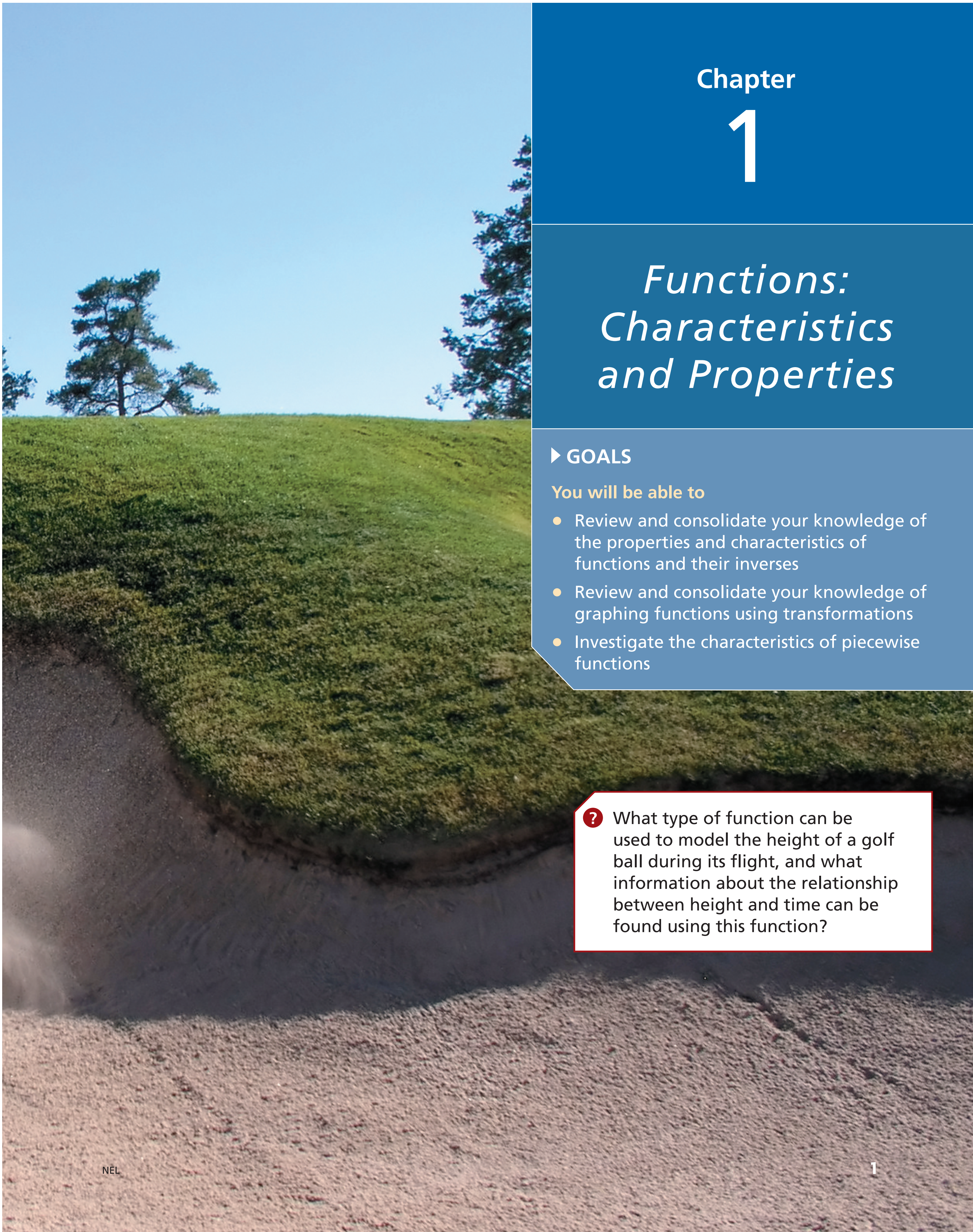


Chapter 1

Functions: Characteristics and Properties

GOALS

You will be able to

- Review and consolidate your knowledge of the properties and characteristics of functions and their inverses
- Review and consolidate your knowledge of graphing functions using transformations
- Investigate the characteristics of piecewise functions

What type of function can be used to model the height of a golf ball during its flight, and what information about the relationship between height and time can be found using this function?

1 Getting Started

Study **Aid**

- For help, see the Review of Essential Skills found at the Nelson Advanced Functions website.

Question	Appendix
2	R-3
3	R-8, R-12

SKILLS AND CONCEPTS You Need

1. Evaluate $f(x) = x^2 + 3x - 4$ for each of the following values.

a) $f(2)$ **b)** $f(-1)$ **c)** $f\left(\frac{1}{4}\right)$ **d)** $f(a + 1)$

2. Factor each of the following expressions.

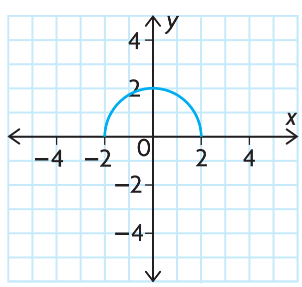
a) $x^2 + 2xy + y^2$ **c)** $(x + y)^2 - 64$
b) $5x^2 - 16x + 3$ **d)** $ax + bx - ay - by$

3. State the **transformations** that are applied to each **parent function**, resulting in the given transformed function. Sketch the graphs of the parent function and transformed function.

a) $f(x) = x^2, y = f(x - 3) + 2$ **c)** $g(x) = \sin x, y = -2g(0.5x)$
b) $f(x) = 2^x, y = f(x - 1) + 2$ **d)** $g(x) = \sqrt{x}, y = -2g(2x)$

4. State the **domain** and **range** of each function.

a)



b) $f(x) = x^2 - 6x - 10$

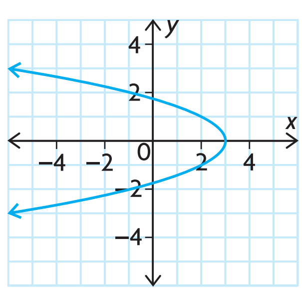
c) $y = \frac{1}{x}$

d) $y = 3 \sin x$

e) $g(x) = 10^x$

5. Which of the following represent functions? Explain.

a)



b) $y = 2(x - 1)^2 + 3$

c) $y = \pm\sqrt{x - 4}$

d) $y = 2^x - 4$

e) $y = \cos(2(x - 30°) + 1)$

6. Consider the **relation** $y = x^3$.

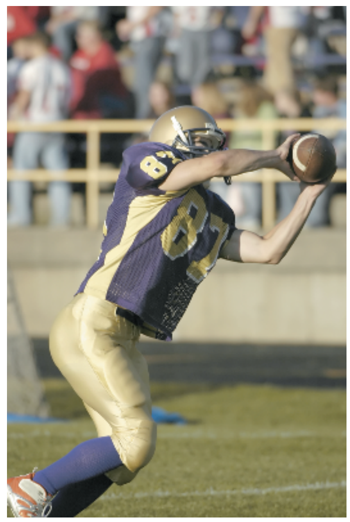
a) If $(2, n)$ is a point on its graph, determine the value of n.
b) If $(m, 20)$ is a point on its graph, determine m correct to two decimal places.

7. A function can be described or defined in many ways. List these different ways, and explain how each can be used to determine whether a relation is a function.

APPLYING What You Know

Modelling the Height of a Football

During a football game, a football is thrown by a quarterback who is 2 m tall. The football travels through the air for 4 s before it is caught by the wide receiver.



? What function can be used to model the height of the football above the ground over time?

A. Explain why the variables time, t, in seconds and height, $h(t)$, in metres are good choices to model this situation.

B. What is $h(0)$? What does it mean in the context of this situation?

C. What happens at $t = 2$ s?

D. What happens at $t = 4$ s?

E. Explain why each of the following functions is *not* a good model for this situation. Support your claim with reasons and a well-labelled sketch.

  i) $h(t) = -5t(t - 4)$
  ii) $h(t) = -5(t - 4)^2 + 2$
  iii) $h(t) = 5t^2 + 4t - 3$

F. Determine a model that can be used to represent the height of the football, given this additional information:
- The ball reached a maximum height of 22 m.
- The wide receiver who caught the ball is also 2 m tall.

G. Use your model from part F to graph the height of the football over the duration of its flight.

1.1 Functions

YOU WILL NEED

- graph paper
- graphing calculator (optional)

GOAL

Represent and describe functions and their characteristics.

LEARN ABOUT the Math

Jonathan and Tina are building an outdoor skating rink. They have enough materials to make a rectangular rink with an area of about 1800 m^2, and they do not want to purchase any additional materials. They know, from past experience, that a good rink must be approximately 30 m longer than it is wide.

? What dimensions should they use to make their rink?

EXAMPLE 1 Representing a situation using a mathematical model

Determine the dimensions that Jonathan and Tina should use to make their rink.

Solution A: Using an algebraic model

Let x represent the length. Let y represent the width.

$$A = xy$$
$$1800 = xy$$
$$\frac{1800}{x} = y$$

We know the area must be 1800 m^2, so if we let the width be the **independent variable**, we can write an expression for the length.

The width, in terms of x, is $\frac{1800}{x}$.

Let $f(x)$ represent the difference between the length and the width.

$$f(x) = x - \frac{1800}{x},$$

where $f(x) = 30$.

$$x - \frac{1800}{x} = 30$$

Using **function notation**, write an equation for the difference in length and width. The relation is a **function** because each input produces a unique output. In this case the difference or value of the function must be 30.

$$x(x) - x\left(\frac{1800}{x}\right) = x(30)$$

To solve the equation, multiply all the terms in the equation by the lowest common denominator, *x*, to eliminate any rational expressions.

$$x^2 - 1800 = 30x$$
$$x^2 - 30x - 1800 = 0$$
$$(x - 60)(x + 30) = 0$$

This results in a quadratic equation. Rearrange the equation so that it is in the form $ax^2 + bx + c = 0$. Factor the left side.

$$x - 60 = 0 \text{ or } x + 30 = 0$$
$$x = 60 \text{ or } x = -30$$

Solve for each factor. $x = -30$ is outside the domain of the function, since length cannot be negative. This is an inadmissible solution.

The length is 60 m.

$$y = \frac{1800}{60} = 30$$

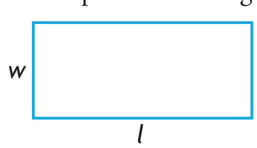
Calculate the width.

The width is 30 m.

The dimensions that are 30 m apart and will produce an area of 1800 m^2 are 60 m × 30 m.

Solution B: Using a numerical model

Let l represent the length. Let w represent the width.



Length is the independent variable.
Its domain is $0 < l < 1800$.
Width is the dependent variable.

$$A = lw$$
$$1800 = lw$$
$$\frac{1800}{l} = w$$

Write an equation for the width in terms of length for a fixed area of 1800 m^2.

Guess 1: $l = 200$

$$w = \frac{1800}{200} = 9$$

Check: $l - w = 200 - 9 \neq 30$

Guess 2: $l = 100$

$$w = \frac{1800}{100} = 18$$

Check: $l - w = 100 - 18 \neq 30$

Use different values for the length to calculate possible widths. Check to see if the difference between the length and width is 30.

Area (m^2)	Length (m)	Width (m)	Length – Width
1800	100	18	82
1800	90	20	70
1800	80	22.5	57.5
1800	70	25.71	44.29
1800	60	30	30
1800	50	36	14
1800	40	45	−5
1800	30	60	−30
1800	20	90	−70

Create a table of values to investigate the difference between the length and the width for a variety of lengths.

The dimensions that are 30 m apart and produce an area of 1800 m^2 are 60 m $\times$ 30 m.

A function can also be represented with a graph. A graph provides a visual display of how the variables in the function are related.

Solution C: Using a graphical model

Let x represent the length. Let y represent the width.

$$A = xy$$
$$1800 = xy$$
$$\frac{1800}{x} = y$$

Using length (x) as the independent variable, write an expression for width (y).

The width, in terms of x, is $\frac{1800}{x}$.

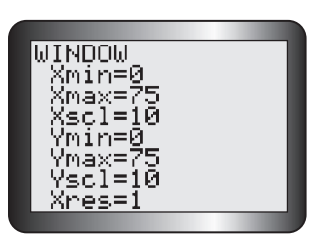
Let $f(x)$ represent the difference between the dimensions.

$$f(x) = x - \frac{1800}{x}$$

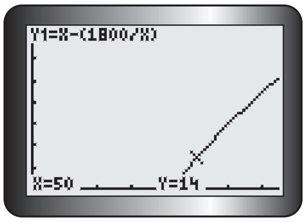
Determine the appropriate window settings to graph $f(x)$ on a graphing calculator.

The value for x (length of rink) will be positive but surely less than 75 m, so we use Xmin = 0 and Xmax = 75.

We use the same settings for the range of f(x), for simplicity.



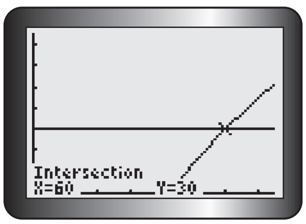
Graph the difference function.

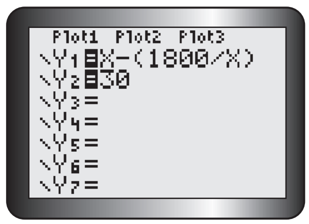


Use the TRACE feature on the graph to investigate points with the ordered pairs (length, length − width) on $f(x)$.

A length of 50 m gives a 14 m difference between the length and the width.

Determine the length that exceeds the width by 30 m.





To determine the length that is 30 m longer than the width, graph $g(x) = 30$ in Y_2 and locate the point of intersection for $g(x)$ and $f(x)$.

The dimensions that are 30 m apart and produce an area of 1800 m^2 are 60 m × 30 m.

Tech Support

For help using the graphing calculator to find points of intersection, see Technical Appendix, T-12.

Reflecting

A. Would the function change if width was used as the independent variable instead of length? Explain.

B. Is it necessary to restrict the domain and range in this problem? Explain.

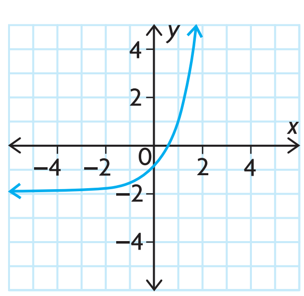
C. Why was it useful to think of the relationship between the length and the width as a function to solve this problem?

APPLY the Math

EXAMPLE 2 Using reasoning to decide whether a relation is a function

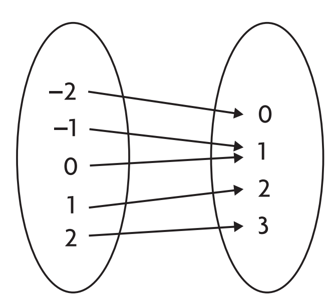
Decide whether each of the following relations is a function. State the domain and range.

a)



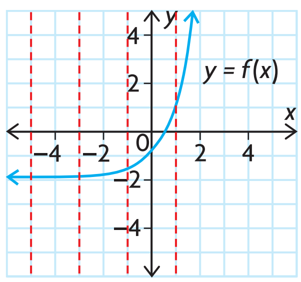
b) $y = \dfrac{1}{x^2}$

c)



Solution

a)



Apply the **vertical line test**. Any vertical line drawn on the graph of a function passes through, at most, a single point. This indicates that each number in the domain corresponds to only one number in the range, which is the condition for the relation to be a function.

The graph represents an **exponential function**.

$D = \{x \in \mathbf{R}\}$
$R = \{y \in \mathbf{R} \mid y > -2\}$

Since the graph of this function has no breaks, or vertical **asymptotes**, and continues indefinitely in both the positive and negative direction, its domain consists of all the **real numbers**.

The function has a **horizontal asymptote** defined by the equation $y = -2$. All its values lie above this horizontal line.

b)

x	−3	−2	−1	0	1	2	3
***f*(*x*)**	$\frac{1}{9}$	$\frac{1}{4}$	1	undefined	1	$\frac{1}{4}$	$\frac{1}{9}$

Create a table of values.

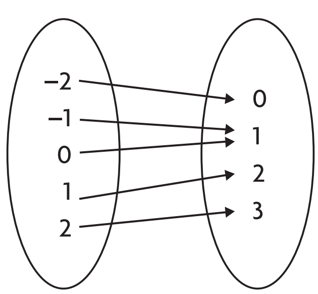
The table indicates that each number in the domain corresponds to only one number in the range.

$f(x) = \frac{1}{x^2}$ is a function.

$D = \{x \in \mathbf{R} \mid x \neq 0\}$
$R = \{y \in \mathbf{R} \mid y > 0\}$

$f(x) = \frac{1}{x^2}$ has a **vertical asymptote** defined by $x = 0$. Its domain consists of all the real numbers, except 0. It has a horizontal asymptote defined by the equation $y = 0$. All its values are positive, since x is squared, so they lie above this horizontal line.

c)



The mapping diagram indicates that each number in the domain corresponds to only one number in the range.

A function can have converging arrows but cannot have diverging arrows in a mapping diagram.

This is a function.

$D = \{-2, -1, 0, 1, 2\}$
$R = \{0, 1, 2, 3\}$

The first oval represents the elements found in the domain. The second oval represents the elements found in the range.

EXAMPLE 3 Using reasoning to determine the domain and range of a function

Naill rides a Ferris wheel that has a diameter of 6 m. The axle of the Ferris wheel is 4 m above the ground. The Ferris wheel takes 90 s to make one complete rotation, and Naill rides for 10 rotations. What are the domain and range of the function that models Naill's height above the ground, in terms of time, while he rides the Ferris wheel?

Solution

$h(t) = a \sin [k(t - d)] + c$

or

$h(t) = a \cos [k(t - d)] + c$

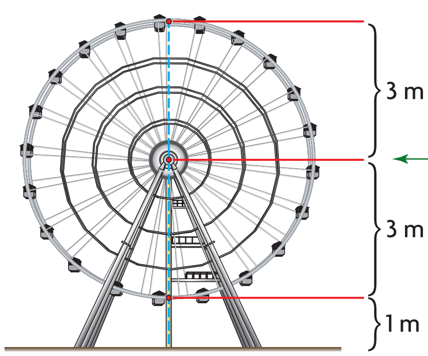
This situation involves circular motion, which can be modelled by a sine or cosine function.

$D = \{t \in \mathbf{R} \mid 0 \le t \le 900\}$

Examine the conditions on the independent variable time to determine the domain. Time cannot be negative, so the lower boundary is 0. The wheel rotates once every 90 s, and Naill rides for 10 complete rotations.

$90 \times 10 = 900$

The upper boundary is 900 s.



$R = \{h(t) \in \mathbf{R} \mid 1 \le h(t) \le 7\}$

Examine the conditions on the dependent variable height to determine the range. The radius of the wheel is 3 m. Since the axle is located 4 m above the ground, the lowest height that Naill can be above the ground is the difference between the height of the axle and the radius of the wheel: $4 - 3 = 1$ m. This is the lower boundary of the range.

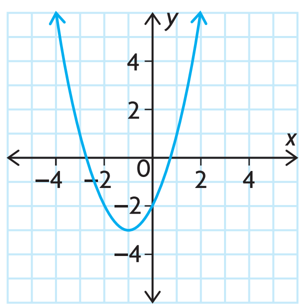
The greatest height he reaches is the sum of the height of the axle and the radius of the wheel: $4 + 3 = 7$ m. This is the upper boundary of the range.

In Summary

Key Ideas

- A function is a relation in which there is a unique output for each input. This means that each value of the independent variable (the domain) must correspond to one, and only one, value of the dependent variable (the range).
- Functions can be represented graphically, numerically, or algebraically.

Graphical Example



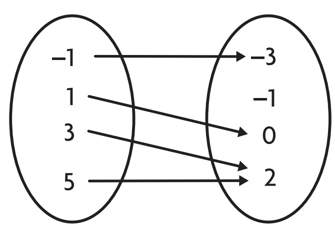
Numerical Examples

Set of ordered pairs:
$\{(1, 3), (3, 5), (-2, 9), (5, 11)\}$

Table of values:

x	y
-2	4
-1	1
0	0
1	1
2	4

Mapping diagram:

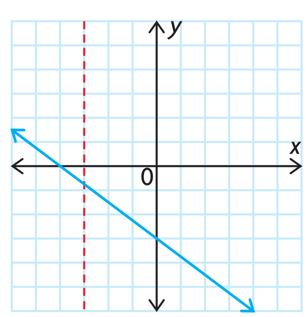


Algebraic Examples

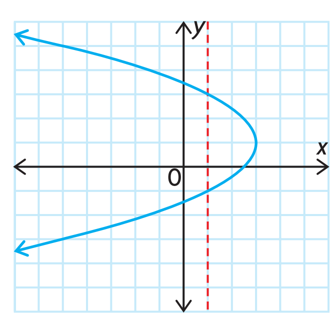
$y = 2 \sin (3x) + 4$
or
$f(x) = 2 \sin (3x) + 4$

Need to Know

- Function notation, $f(x)$, is used to represent the values of the dependent variable in a function, so $y = f(x)$.
- You can use the vertical line test to check whether a graph represents a function. A graph represents a function if every vertical line intersects the graph in, at most, one point. This shows that there is only one element in the range for each element in the domain.
- The domain and range of a function depend on the type of function.
- The domain and range of a function that models a particular situation may need to be restricted, based on the situation. For example, negative values may not have meaning when dealing with variables such as time.



function

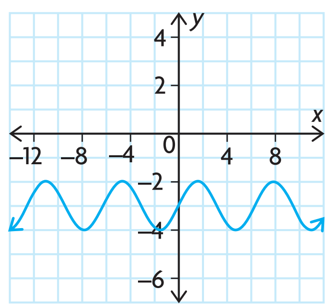


not a function

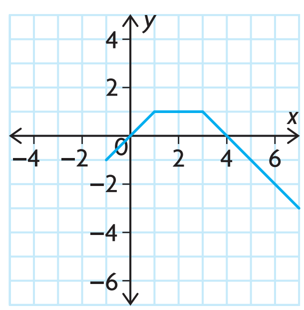
CHECK Your Understanding

1. State the domain and range of each relation. Then determine whether the relation is a function, and justify your answer.

a)



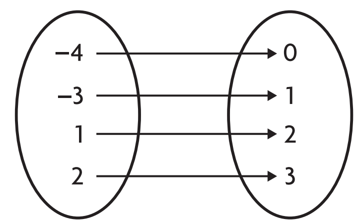
b)



c) $\{(1, 4), (1, 9), (2, 7), (3, -5), (4, 11)\}$

d) $y = 3x - 5$

e)



f) $y = -5x^2$

2. State the domain and range of each relation. Then determine whether the relation is a function, and justify your answer.

a) $y = -2(x + 1)^2 - 3$

b) $y = \dfrac{1}{x + 3}$

c) $y = 2^{-x}$

d) $y = \cos x + 1$

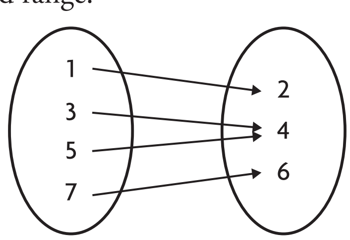
e) $x^2 + y^2 = 9$

f) $y = 2 \sin x$

PRACTISING

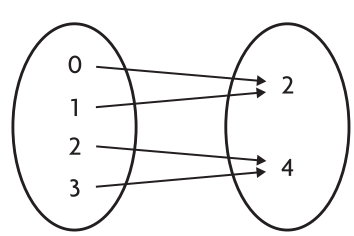
3. Determine whether each relation is a function, and state its domain and range.

a)



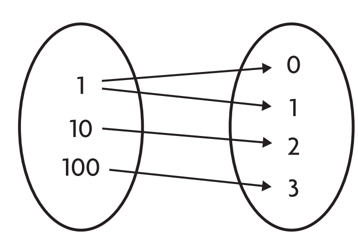
b) $\{(2, 3), (1, 3), (5, 6), (0, -1)\}$

c)



d) $\{(2, 5), (6, 1), (2, 7), (8, 3)\}$

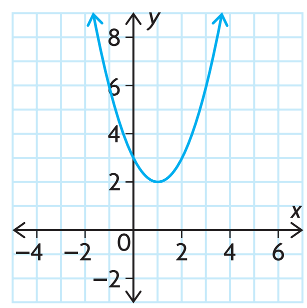
e)



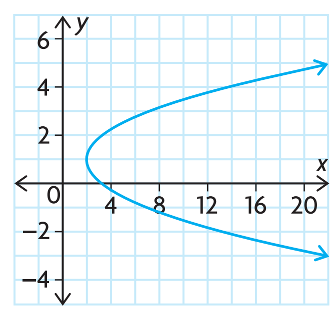
f) $\{(1, 2), (2, 1), (3, 4), (4, 3)\}$

4. Determine whether each relation is a function, and state its domain and range.
K

a)



b)



c) $x^2 = 2y + 1$

d) $x = y^2$

e) $y = \frac{3}{x}$

f) $f(x) = 3x + 1$

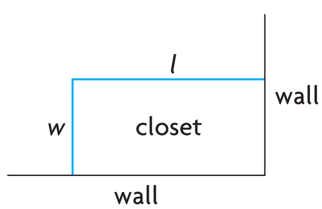
5. Determine the equations that describe the following function rules:

a) The input is 3 less than the output.

b) The output is 5 less than the input multiplied by 2.

c) Subtract 2 from the input and then multiply by 3 to find the output.

d) The sum of the input and output is 5.



6. Martin wants to build an additional closet in a corner of his bedroom. Because the closet will be in a corner, only two new walls need to be built. The total length of the two new walls must be 12 m. Martin wants the length of the closet to be twice as long as the width, as shown in the diagram.

a) Explain why $l = 2w$.

b) Let the function $f(l)$ be the sum of the length and the width. Find the equation for $f(l)$.

c) Graph $y = f(l)$.

d) Find the desired length and width.

7. The following table gives Tina's height above the ground while riding a Ferris wheel, in relation to the time she was riding it.
A

Time (s)	0	20	40	60	80	100	120	140	160	180	200	220	240
Height (m)	5	10	5	0	5	10	5	0	5	10	5	0	5

a) Draw a graph of the relation, using time as the independent variable and height as the dependent variable.

b) What is the domain?

c) What is the range?

d) Is this relation a function? Justify your answer.

e) Another student sketched a graph, but used height as the independent variable. What does this graph look like?

f) Is the relation in part e) a function? Justify your answer.

8. Consider what happens to a relation when the coordinates of all its ordered pairs are switched.
   a) Give an example of a function that is still a function when its coordinates are switched.
   b) Give an example of a function that is no longer a function when its coordinates are switched.
   c) Give an example of a relation that is not a function, but becomes a function when its coordinates are switched.

9. Explain why a relation that fails the vertical line test is not a function.

10. Consider the relation between x and y that consists of all points (x, y) such that the distance from (x, y) to the origin is 5.
   a) Is (4, 3) in the relation? Explain.
   b) Is (1, 5) in the relation? Explain.
   c) Is the relation a function? Explain.

11. The table below lists all the ordered pairs that belong to the function $g(x)$.

x	0	1	2	3	4	5
$g(x)$	3	4	7	12	19	28

   a) Determine an equation for $g(x)$.
   b) Does $g(3) - g(2) = g(3 - 2)$? Explain.

12. **T** The factors of 4 are 1, 2, and 4. The sum of the factors is $1 + 2 + 4 = 7$. The sum of the factors is called the sigma function. Therefore, $f(4) = 7$.
   a) Find $f(6)$, $f(7)$, and $f(8)$.
   b) Is $f(15) = f(3) \times f(5)$?
   c) Is $f(12) = f(3) \times f(4)$?
   d) Are there others that will work?

13. **C** Make a concept map to show what you have learned about functions. Put "FUNCTION" in the centre of your concept map, and include the following words:

algebraic model, graphical model, numerical model, dependent variable, independent variable, range, domain, mapping model, vertical line test, function notation

Communication *Tip*

A concept map is a type of web diagram used for exploring knowledge and gathering and sharing information. A concept map consists of cells that contain a concept, item, or question and links. The links are labelled and denote direction with an arrow symbol. The labelled links explain the relationship between the cells. The arrow describes the direction of the relationship and reads like a sentence.

Extending

14. Consider the relations $x^2 + y^2 = 25$ and $y = \sqrt{25 - x^2}$. Draw the graphs of these relations, and determine whether each relation is a function. State the domain and range of each relation.

15. You already know that y is a function of x if and only if the graph passes the vertical line test. When is x a function of y? Explain.

1.2 Exploring Absolute Value

YOU WILL NEED

- graph paper
- graphing calculator

GOAL

Discover the properties of the absolute value function.

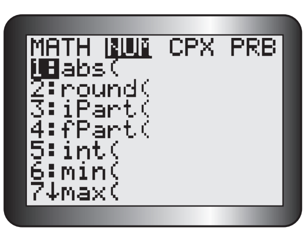
EXPLORE the Math

An average person's blood pressure is dependent on their age and gender. For example, the average systolic blood pressure, P_n, for a 17-year-old girl is about 127 mm Hg. (The symbol mm Hg stands for millimetres of mercury, which is a unit of measure for blood pressure.) The average systolic blood pressure for a 17-year-old boy is about 134 mm Hg.

When doctors measure blood pressure, they compare the blood pressure to the average blood pressure for people in the same age and gender group. This comparison, P_d, is calculated using the formula $P_d = |P - P_n|$, where P is the blood pressure reading and P_n is the average reading for people in the same age and gender group.

Tech *Support*

To use the **absolute value** command on a graphing calculator, press MATH and scroll right to NUM. Then press ENTER.



? How can the blood pressure readings of a group of people be compared?

A. Jim is a 17-year-old boy whose most recent blood pressure reading was 142 mm Hg. Calculate P_d for Jim.

B. Joe is a 17-year-old boy whose most recent blood pressure reading was 126 mm Hg. Calculate P_d for Joe.

C. Compare the values of $P - P_n$ and $|P - P_n|$ that were used to determine P_d for each boy. What do you notice?

D. Complete the following table by calculating the values of P_d for the given blood pressure readings for 17-year-old boys.

Blood Pressure Reading, *P*	95	100	105	110	115	120	125	130	135	140	145	150	155	160
P_d														

E. Draw a **scatter plot** of P_d as a function of blood pressure, P.

F. Describe these characteristics of your graph:

i) domain
ii) range
iii) zeros
iv) existence of any asymptotes
v) shape of the graph
vi) intervals of the domain in which the values of the function P_d are increasing and decreasing.
vii) behaviour of the values of the function P_d as P becomes larger and smaller

Reflecting

G. Why might you predict the range of your graph to be greater than or equal to zero?

H. What other function with domain greater than P_n could you have used to plot the right side of your graph? Why does this make sense?

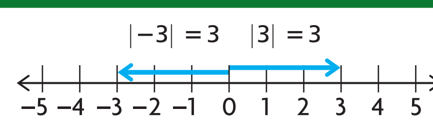
I. What other function with domain less than P_n could you have used to plot the left side of your graph? Why does this make sense?

J. How will the graph of $y = |x|$ compare with the graph of $P_d = |P - P_n|$, if P_d is the y-coordinate and P is the x-coordinate? Use the characteristics you listed in part F to make your comparison.

In Summary

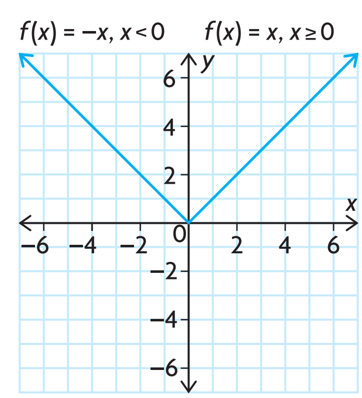
Key Idea

- $f(x) = |x|$ is the absolute value function. On a number line, this function describes the distance, $f(x)$, of any number x from the origin.



Need to Know

- For the function $f(x) = |x|$,
  - there is one zero located at the origin
  - the graph is comprised of two linear functions and is defined as follows:
    $$f(x) = \begin{cases} x, \text{ if } x \geq 0 \\ -x, \text{ if } x < 0 \end{cases}$$
  - the graph is symmetric about the y-axis
  - as x approaches large positive values, y approaches large positive values
  - as x approaches large negative values, y approaches large positive values
  - the absolute value function has domain $\{x \in \mathbf{R}\}$ and range $\{y \in \mathbf{R} | y \geq 0\}$
  - every input in an absolute value returns an output that is non-negative



FURTHER *Your Understanding*

1. Arrange these values in order, from least to greatest:
$|-5|, |20|, |-15|, |12|, |-25|$

2. Evaluate.
   - **a)** $|-22|$
   - **b)** $-|-35|$
   - **c)** $|-5 - 13|$
   - **d)** $|4 - 7| + |-10 + 2|$
   - **e)** $\dfrac{|-8|}{-4}$
   - **f)** $\dfrac{|-22|}{|-11|} + \dfrac{-16}{|-4|}$

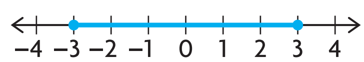
3. Express using absolute value notation.
   - **a)** $x < -3$ or $x > 3$
   - **b)** $-8 \leq x \leq 8$
   - **c)** $x \leq -1$ or $x \geq 1$
   - **d)** $x \neq \pm 5$

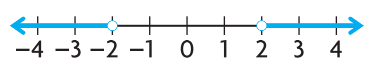
Communication ***Tip***

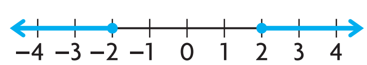
To show that a number is not included in the solution set, use an open dot at this value. A solid dot shows that this value is included in the solution set.

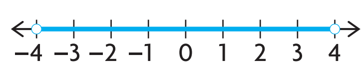
4. Graph on a number line.
   - **a)** $|x| < 8$
   - **b)** $|x| \geq 16$
   - **c)** $|x| \leq -4$
   - **d)** $|x| > -7$

5. Rewrite using absolute value notation.

a)


b)


c)


d)


6. Graph $f(x) = |x - 8|$ and $g(x) = |-x + 8|$.
   - **a)** What do you notice?
   - **b)** How could you have predicted this?

7. Graph the following functions.
   - **a)** $f(x) = |x - 2|$
   - **b)** $f(x) = |x| + 2$
   - **c)** $f(x) = |x + 2|$
   - **d)** $f(x) = |x| - 2$

8. Compare the graphs you drew in question 7. How could you use transformations to describe the graph of $f(x) = |x + 3| - 4$?

9. Predict what the graph of $f(x) = |2x + 1|$ will look like. Verify your prediction using graphing technology.

10. Predict what the graph of $f(x) = 3 - |2x - 5|$ will look like. Verify your prediction using graphing technology.

1.3 Properties of Graphs of Functions

GOAL

Compare and contrast the properties of various types of functions.

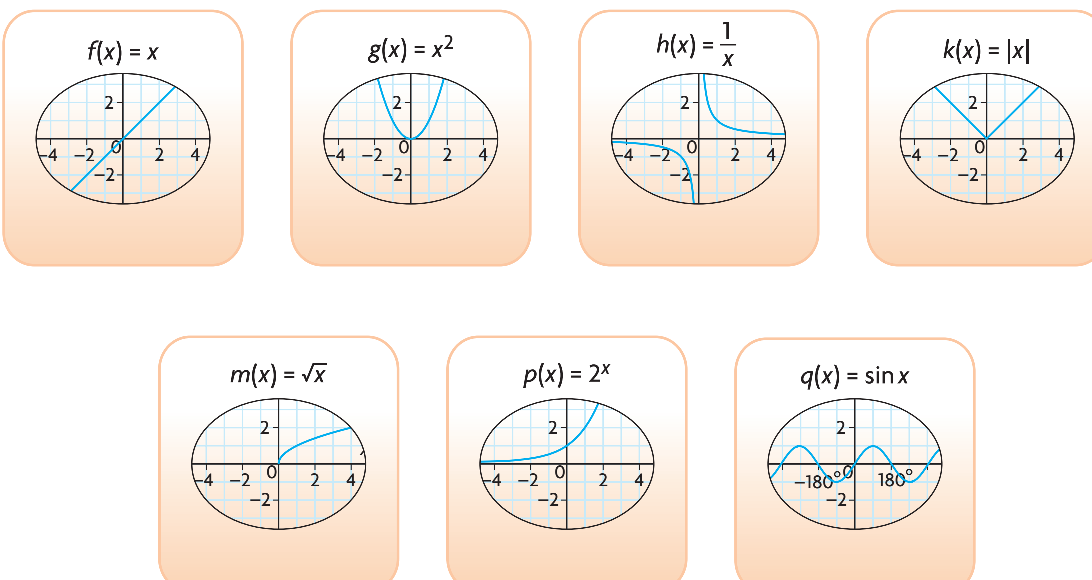
YOU WILL NEED

- graphing calculator

INVESTIGATE the Math

Two students created a game that they called "Which function am I?" In this game, players turn over cards that are placed face down and match the characteristics and properties with the correct functions. The winner is the player who has the most pairs at the end of the game.

The students have studied the following parent functions:



? Which criteria could the students use to differentiate between these different types of functions?

A. Graph each of these parent functions on a graphing calculator, and sketch its graph. State the domain and range of each function, and determine its zeros and y-intercepts.

B. Determine the intervals of increase and the intervals of decrease for each of the parent functions.

interval of increase

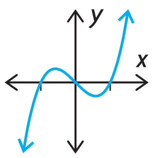
the interval(s) within a function's domain, where the y-values of the function get larger, moving from left to right

interval of decrease

the interval(s) within a function's domain, where the y-values of the function get smaller, moving from left to right

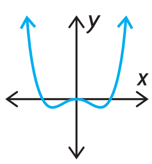
odd function

any function that has rotational symmetry about the origin; algebraically, all odd functions have the property $f(-x) = -f(x)$



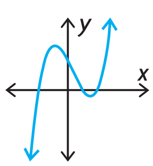
even function

any function that is symmetric about the y-axis; algebraically, all even functions have the property $f(-x) = f(x)$



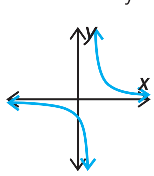
continuous function

any function that does not contain any holes or breaks over its entire domain



discontinuity

a break in the graph of a function is called a point of discontinuity



C. State whether each parent function is an odd function, an even function, or neither.

D. Do any of the functions have vertical or horizontal asymptotes? If so, what are the equations of these asymptotes?

E. Which graphs are continuous? Which have discontinuities?

F. Complete the following statements to describe the end behaviour of each parent function.

a) As x increases to large positive values, y . . .

b) As x decreases to large negative values, y . . .

Communication *Tip*

It is often convenient to use the symbol for infinity, ∞, and the following notation to write the end behaviour of a function:

- For "As x increases to large positive values, y . . . ," write "As $x \rightarrow \infty$, $y \rightarrow$ . . ."
- For "As x decreases to large negative values, y . . . ," write "As $x \rightarrow -\infty$, $y \rightarrow$ . . ."

G. Summarize your findings.

Reflecting

H. Which of the parent functions can be distinguished by their domain? Which can be distinguished by their range? Which can be distinguished by their zeros?

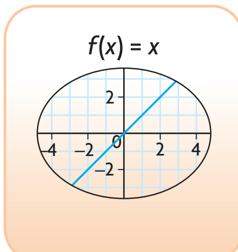
I. An increasing function is one in which the function's values increase from left to right over its entire domain. A decreasing function is one in which the function's values decrease from left to right over its entire domain. Which of the parent functions are increasing functions? Which are decreasing functions?

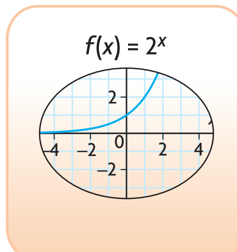
J. Which properties of each function would make the function easy to identify from a description of it?

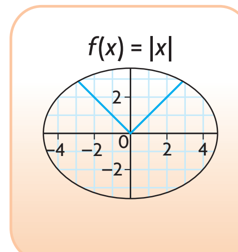
APPLY the Math

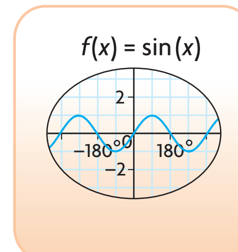
EXAMPLE 1 Connecting the graph of a function with its characteristics

Match each parent function card with a characteristic of its graph. Each card may only be used for one parent function.



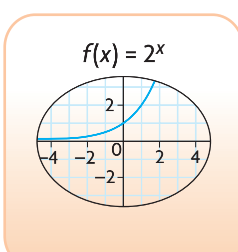






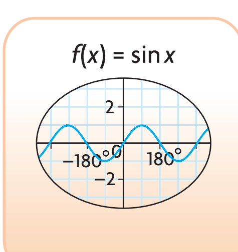
Range: $\{y \in \mathbb{R} \mid y \geq 0\}$	Domain: $\{x \in \mathbb{R}\}$	Infinite Number of Zeros	As $x \to -\infty$, $y \to 0$.

Solution



As $x \to -\infty$, $y \to 0$.

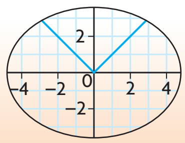
This property describes the end behaviour: as x becomes negatively large, y approaches zero. The function must have a horizontal asymptote defined by $y = 0$. The function must be $y = 2^x$.



Infinite Number of Zeros

The sine function is periodic and continues infinitely, intersecting the x-axis an infinite number of times.

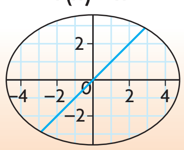
$f(x) = |x|$



Range:
$\{y \in \mathbf{R} \mid y \geq 0\}$

The range $\{y \in \mathbf{R} \mid y \geq 0\}$ indicates that all y-values of the function must be non-negative. This is true for both $f(x) = 2^x$ and $f(x) = |x|$. However, $f(x) = 2^x$ has the x-axis as its horizontal asymptote, so $y \neq 0$. Choose $f(x) = |x|$.

$f(x) = x$



Domain:
$\{x \in \mathbf{R}\}$

The remaining function, $f(x) = x$, matches the property $\{x \in \mathbf{R}\}$.

If you are given some characteristics of a function, you may be able to determine the equation of the function.

EXAMPLE 2 Using reasoning to determine the equation of a parent function

State which of the parent functions in this lesson have the following characteristics:

a) Domain $= \{x \in \mathbf{R}\}$

b) Range $= \{y \in \mathbf{R} \mid -1 \leq y \leq 1\}$

Solution

a) Domain $= \{x \in \mathbf{R}\}$

$$f(x) = x$$
$$g(x) = x^2$$
$$\cancel{h(x) = \frac{1}{x}} \text{ (Domain } = \{x \in \mathbf{R} \mid x \neq 0\})$$
$$k(x) = |x|$$
$$\cancel{m(x) = \sqrt{x}} \text{ (Domain } = \{x \in \mathbf{R} \mid x \geq 0\})$$
$$p(x) = 2^x$$
$$q(x) = \sin x$$

There are five parent functions that match this characteristic and two that do not.

b) Range $= \{y \in \mathbf{R} \mid -1 \leq y \leq 1\}$

$$\cancel{f(x) = x} \text{ (Range } = \{y \in \mathbf{R}\})$$
$$\cancel{g(x) = x^2} \text{ (Range } = \{y \in \mathbf{R} \mid y \geq 0\})$$
$$\cancel{k(x) = |x|} \text{ (Range } = \{y \in \mathbf{R} \mid y \geq 0\})$$
$$\cancel{p(x) = 2^x} \text{ (Range } = \{y \in \mathbf{R} \mid y \geq 0\})$$
$$\cancel{q(x) = \sin x}$$

Of these five functions, only the sine function has the range $\{y \in \mathbf{R} \mid -1 \leq y \leq 1\}$.

Visualizing what the graph of a function looks like can help you remember some of the characteristics of the function.

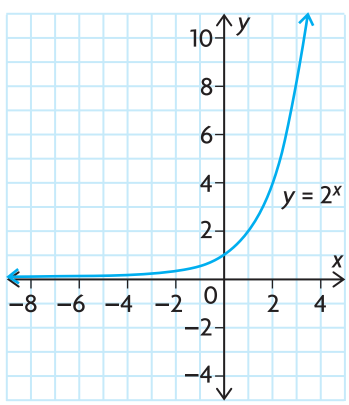
EXAMPLE 3 Connecting the characteristics of a function with its equation

Which of the following are characteristics of the parent function $p(x) = 2^x$? Justify your reasoning.

a) The graph is decreasing for all values in the domain of $p(x)$.
b) The graph is continuous for all values in the domain of $p(x)$.
c) The function $p(x)$ is an even function.
d) The function $p(x)$ has no zeros.

Solution

$p(x) = 2^x$



The function $p(x)$ is an exponential function with a base that is greater than 1.

This type of function is increasing for all values in its domain.

a) This function is increasing for all values in the domain of $p(x)$.

b) The graph is continuous for all values in the domain of $p(x)$.

This function has no breaks.

c) The function $p(x)$ is not an even function.

This type of function is not symmetric about the y-axis. $f(-x) = 2^{-x}$. This substitution does not result in $f(x)$.

d) The function $p(x)$ has no zeros.

As x approaches negative infinity, the graph gets arbitrarily close to the x-axis but does not intersect it.

Only b) and d) are characteristics of $p(x)$.

EXAMPLE 4 Connecting the characteristics of a function with its equation and its graph

Determine a possible transformed parent function that has the following characteristics, and sketch the function:

- $D = \{x \in \mathbf{R}\}$
- $R = \{y \in \mathbf{R} \mid y \geq -2\}$
- decreasing on the interval $(-\infty, 0)$
- increasing on the interval $(0, \infty)$

Communication *Tip*

The interval $(-\infty, 0)$ is described using interval notation and is equivalent to $x < 0$ in set notation. The use of round brackets in interval notation indicates that the endpoint is not included in the interval. The use of square brackets in interval notation indicates that the endpoint is included in the interval. For example, $[-3, 5)$ is equivalent to $-3 \leq x < 5$.

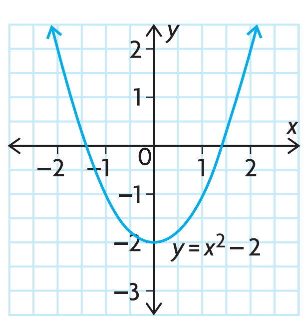
Solution

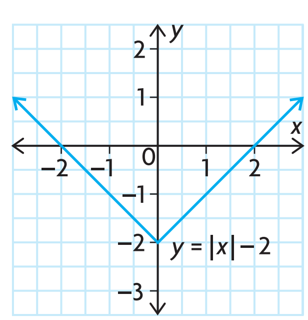
$\cancel{f(x) = x}$
$g(x) = x^2$
$k(x) = |x|$
$\cancel{p(x) = 2^x}$
$\cancel{q(x) = \sin x}$

List the functions that have domain $\{x \in \mathbf{R}\}$. Eliminate the functions that cannot have the range $\{y \in \mathbf{R} \mid y \geq -2\}$. Each of the remaining functions can be translated down two units to have this range.

Function	Intervals of Increase	Intervals of Decrease
$g(x) = x^2$	$(0, \infty)$	$(-\infty, 0)$
$k(x) = \lvert x \rvert$	$(0, \infty)$	$(-\infty, 0)$

State the intervals of increase and decrease for the two remaining functions. Check to see if these intervals match the given conditions. There are two possible parent functions that have the given characteristics.





Sketch the graph of each parent function shifted 2 units down.

In Summary

Key Idea

Functions can be categorized based on their graphical characteristics:

- domain and range
- *x*-intercepts and *y*-intercepts
- continuity and discontinuity
- intervals of increase and decrease
- symmetry (even/odd)
- end behaviour

Need to Know

- Given a set of graphical characteristics, the type of function that has these characteristics can be determined by eliminating those that do not have these characteristics.
- Some characteristics are more helpful than others when determining the type of function.

CHECK Your Understanding

1. Which graphical characteristic is the least helpful for differentiating among the parent functions? Why?

2. Which graphical characteristic is the most helpful for differentiating among the parent functions? Why?

3. One of the seven parent functions examined in this lesson is transformed to yield a graph with these characteristics:

- $D = \{x \in \mathbf{R}\}$
- $R = \{y \in \mathbf{R} \mid y > 2\}$
- As $x \to -\infty$, $y \to 2$.

What is the equation of the transformed function?

PRACTISING

4. **K** For each pair of functions, give a characteristic that the two functions have in common and a characteristic that distinguishes between them.

a) $f(x) = \frac{1}{x}$ and $g(x) = x$

b) $f(x) = \sin x$ and $g(x) = x$

c) $f(x) = x$ and $g(x) = x^2$

d) $f(x) = 2^x$ and $g(x) = |x|$

5. For each function, determine $f(-x)$ and $-f(-x)$ and compare it with $f(x)$. Use this to decide whether each function is even, odd, or neither.

a) $f(x) = x^2 - 4$

b) $f(x) = \sin x + x$

c) $f(x) = \frac{1}{x} - x$

d) $f(x) = 2x^3 + x$

e) $f(x) = 2x^2 - x$

f) $f(x) = |2x + 3|$

6. Determine a possible parent function that could serve as a model for each of the following situations, and explain your choice.
A

a) The number of marks away from the class average that a student's test score is

b) The height of a person above the ground during several rotations of a Ferris wheel

c) The population of Earth throughout time

d) The amount of total money saved if you put aside exactly one dollar every day

7. Identify a parent function whose graph has the given characteristics.

a) The domain is not all real numbers, and $f(0) = 0$.

b) The graph has an infinite number of zeros.

c) The graph is even and has no sharp corners.

d) As x gets negatively large, so does y. As x gets positively large, so does y.

8. Each of the following situations involves a parent function whose graph has been translated. Draw a possible graph that fits the situation.

a) The domain is $\{x \in \mathbf{R}\}$, the interval of increase is $(-\infty, \infty)$, and the range is $\{f(x) \in \mathbf{R} \mid f(x) > -3\}$.

b) The range is $\{g(x) \in \mathbf{R} \mid 2 \leq g(x) \leq 4\}$.

c) The domain is $\{x \in \mathbf{R} \mid x \neq 5\}$, and the range is $\{h(x) \in \mathbf{R} \mid h(x) \neq -3\}$.

9. Sketch a possible graph of a function that has the following characteristics:

- $f(0) = -1.5$
- $f(1) = 2$
- There is a vertical asymptote at $x = -1$.
- As x gets positively large, y gets positively large.
- As x gets negatively large, y approaches zero.

10. **a)** $f(x)$ is a quadratic function. The graph of $f(x)$ decreases on the interval $(-\infty, -2)$ and increases on the interval $(2, \infty)$. It has a y-intercept at $(0, 4)$. What is a possible equation for $f(x)$?
T

b) Is there only one quadratic function, $f(x)$, that has the characteristics given in part a)?

c) If $f(x)$ is an absolute value function that has the characteristics given in part a), is there only one such function? Explain.

11. $f(x) = x^2$ and $g(x) = |x|$ are similar functions. How might you describe the difference between the two graphs to a classmate, so that your classmate can tell them apart?

12. Copy and complete the following table. In your table, highlight the graphical characteristics that are unique to each function and could be used to distinguish it easily from other parent functions.

Parent Function	$f(x) = x$	$g(x) = x^2$	$h(x) = \frac{1}{x}$	$k(x) = \lvert x \rvert$	$m(x) = \sqrt{x}$	$p(x) = 2^x$	$r(x) = \sin x$
Sketch							
Domain							
Range							
Intervals of Increase							
Intervals of Decrease							
Location of Discontinuities and Asymptotes							
Zeros							
y-Intercepts							
Symmetry							
End Behaviours							

13. C Linear, quadratic, reciprocal, absolute value, square root, exponential, and sine functions are examples of different types of functions, with different properties and characteristics. Why do you think it is useful to name these different types of functions?

Extending

14. Consider the parent function $f(x) = x^3$. Graph $f(x)$, and compare and contrast this function with the parent functions you have learned about in this lesson.

15. Explain why it is not necessary to have $h(x) = \cos(x)$ defined as a parent function.

16. Suppose that $g(x) = |x|$ is translated around the coordinate plane. How many zeros can its graph have? Discuss all possibilities, and give an example of each.

1 Mid-Chapter Review

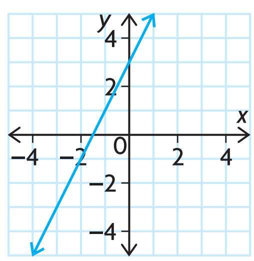
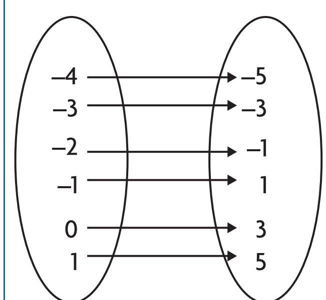
FREQUENTLY ASKED Questions

Study *Aid*

- See Lesson 1.1, Examples 1 and 2.
- Try Mid-Chapter Review Questions 1, 2, and 3.

Q: What is a function, and which of its representations is the best for solving problems and making predictions?

A: A function is a relation between two variables, in which each input has a unique output. Functions can be represented using words, graphs, numbers, and algebra.

<table>
<tr><th>Word Example</th><th>Graphical Example</th><th colspan="2">Numerical Example</th><th>Algebraic Example</th></tr>
<tr>
<td>One number is three more than twice another number.</td>
<td>
</td>
<td>Table of values:
<table>
<tr><th>x</th><th>y</th></tr>
<tr><td>−4</td><td>−5</td></tr>
<tr><td>−3</td><td>−3</td></tr>
<tr><td>−2</td><td>−1</td></tr>
<tr><td>−1</td><td>1</td></tr>
<tr><td>0</td><td>3</td></tr>
<tr><td>1</td><td>5</td></tr>
</table></td>
<td>Mapping diagram:

</td>
<td>$f(x) = 2x + 3$</td>
</tr>
</table>

The algebraic model is the most useful and most accurate. If you know the value of one variable, you can substitute this value into the function to create an equation, which can then be solved using an appropriate strategy. This leads to an accurate answer. Both numerical and graphical models are limited in their use because they represent the function for only small intervals of the domain and range. When using a graphical model, it may be necessary to interpolate or extrapolate. This can lead to approximate answers.

Study *Aid*

- See Lesson 1.2.
- Try Mid-Chapter Review Questions 4 and 5.

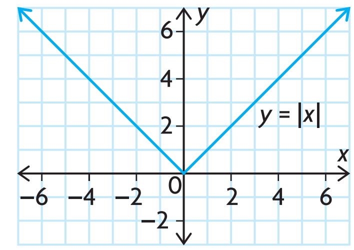
Q: What is the absolute value function, and what are the characteristics of its graph?

A: The absolute value function is $f(x) = |x|$. On a number line, $|x|$ is the distance of any value, x, from the origin. The absolute value function consists of two linear pieces, each defined by a different equation:

$$f(x) = \begin{cases} x, \text{ if } x \geq 0 \\ -x, \text{ if } x < 0 \end{cases}$$

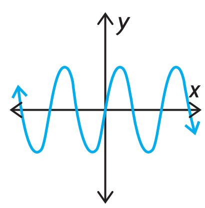
This function has the following characteristics:

- x-intercept: $x = 0$
- y-intercept: $y = 0$
- domain: $D = \{x \in \mathbf{R}\}$; range: $\mathbf{R} = \{y \in \mathbf{R} \mid y \geq 0\}$
- interval of decrease: $(-\infty, 0)$; interval of increase: $(0, \infty)$
- end behaviour: As $x \to \infty$, $y \to \infty$; as $x \to -\infty$, $y \to \infty$.

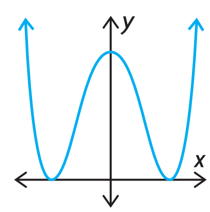


Q: What is the difference between an odd function and an even function, and how are the parent functions differentiated by this characteristic?

A: The graph of an odd function has rotational symmetry about the origin.



The graph of an even function is symmetric about the y-axis.



To test algebraically whether a function is odd or even, substitute $-x$ for x and simplify:
- If $f(-x) = -f(x)$, then the function is odd.
- If $f(-x) = f(x)$, then the function is even.

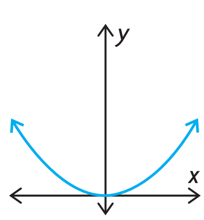
Odd Parent Functions: $f(x) = x, f(x) = \frac{1}{x}, f(x) = \sin x$

Even Parent Functions: $f(x) = x^2, f(x) = |x|, f(x) = \cos x$

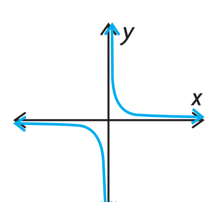
Study Aid
- See Lesson 1.3, Examples 3 and 4.
- Try Mid-Chapter Review Questions 6, 7, and 8.

Q: What is a discontinuity, and what is a continuous function?

A: A discontinuity is a break in the graph of a function. A function is continuous if it has no discontinuities; that is, no holes or breaks in its graph over its entire domain.



a continuous function



The function $y = \frac{1}{x}$ has a discontinuity at $x = 0$.

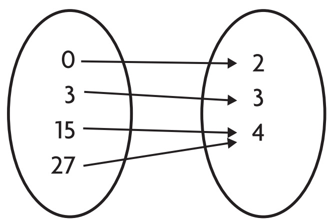
Study Aid
- See Lesson 1.3.
- Try Mid-Chapter Review Question 9.

PRACTICE *Questions*

Lesson 1.1

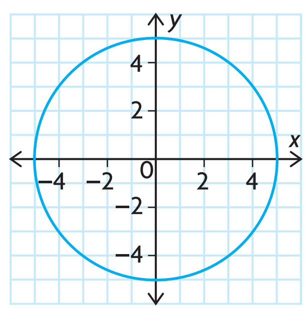
1. Determine whether each relation is a function, and state its domain and range.

a)



b) $y = 2x + 3$

c)



d) $\{(2, 7), (1, 3), (2, 6), (10, -1)\}$

2. The height of a bungee jumper above the ground is modelled by the following data.

Time (s)	0	1	2	3	4	5	6	7	8	9	10
Height (m)	50	40	30	20	10	20	30	40	45	35	25

a) Is the relationship between height and time a function? Explain.
b) What is the domain?
c) What is the range?

3. Determine the domain and range for each of the following and state whether it is a function:

a) $f(x) = 3x + 1$ **c)** $y = \sqrt{5 - x}$
b) $x^2 + y^2 = 9$ **d)** $x^2 - y = 2$

Lesson 1.2

4. Arrange the following values in order, from least to greatest:
$|-3|, -|3|, |5|, |-4|, |0|$

5. Sketch the graph of each function.

a) $f(x) = |x| + 3$ **c)** $f(x) = |-2x|$
b) $f(x) = |x| - 2$ **d)** $f(x) = |0.5x|$

Lesson 1.3

6. Determine a parent function that matches each set of characteristics.

a) The graph is neither even nor odd, and as $x \to \infty, y \to \infty$.
b) $(-\infty, 0)$ and $(0, \infty)$ are both intervals of decrease.
c) The domain is $[0, \infty)$.

7. Determine algebraically if each function is even, odd, or neither.

a) $f(x) = |2x|$ **c)** $f(x) = x + 4$
b) $f(x) = (-x)^2$ **d)** $f(x) = 4x^5 + 3x^3 - 1$

8. Each set of characteristics describes a parent function that has been shifted. Draw a possible graph, and state whether the graph is continuous.

a) There is a vertical asymptote at $x = 1$ and a horizontal asymptote at $y = 3$.
b) The range is $\{f(x) \in \mathbf{R} | -3 \le f(x) \le -1\}$.
c) The interval of increase is $(-\infty, \infty)$, and there is a horizontal asymptote at $y = -10$.

9. Sketch a graph that has the following characteristics:

- The function is odd.
- The function is continuous.
- The function has zeros at $x = -3$, 0, and 3.
- The function is increasing on the intervals $x \in (-\infty, -2)$ or $x \in (2, \infty)$.
- The function is decreasing on the interval $x \in (-2, 2)$.

1.4 Sketching Graphs of Functions

GOAL

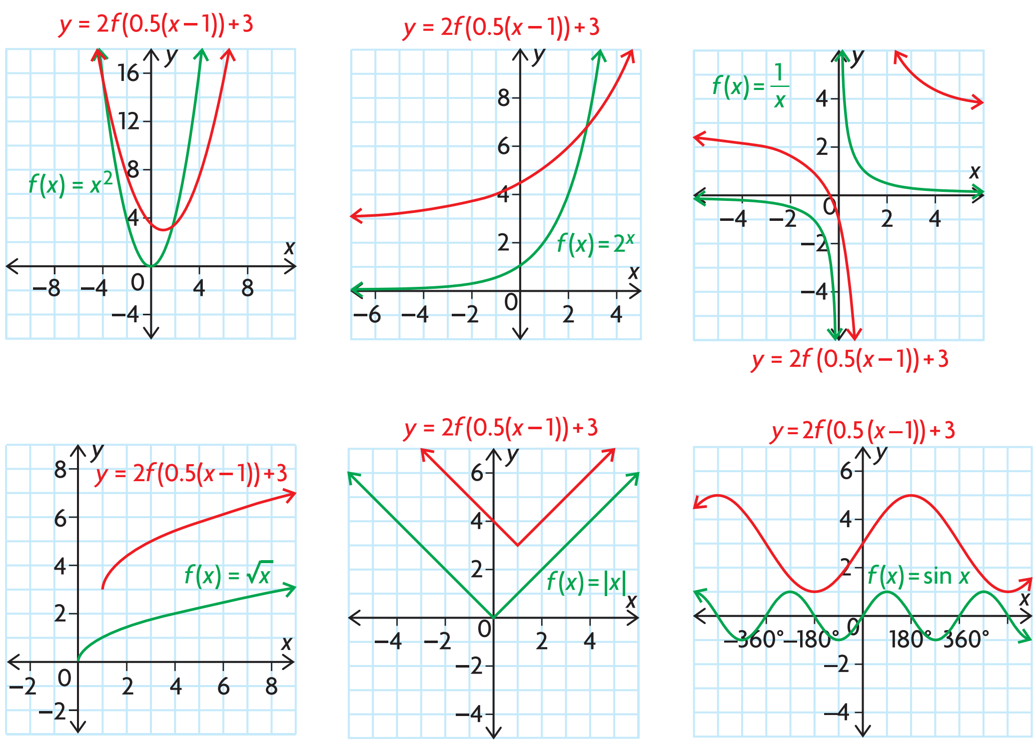
Apply transformations to parent functions, and use the most efficient methods to sketch the graphs of the functions.

YOU WILL NEED

- graph paper
- graphing calculator

INVESTIGATE the Math

The same transformations have been applied to six different parent functions, as shown below.



? How do the transformations defined by $y = 2f(0.5(x - 1)) + 3$ affect the characteristics of each parent function?

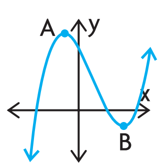
A. Identify the parent function for each graph.

B. Copy and complete the following table for each parent function.

Parent Fuction	$y = x^2$	$y = \frac{1}{x}$	$y = \lvert x \rvert$	$y = 2^x$	$y = \sqrt{x}$	$y = \sin x$
Domain						
Range						
Intervals of Increase						
Intervals of Decrease						
Turning Points						

turning point

a point on a curve where the function changes from increasing to decreasing, or vice versa; for example, A and B are turning points on the following curve



C. Identify the transformations (in the correct order) that were performed on each parent function to arrive at the transformed function.

D. State the transformation(s) that affected each of the following characteristics for each of the parent functions in the table above.

i) domain
ii) range
iii) intervals of increase/decrease
iv) turning points
v) the equation(s) of any vertical asymptotes
vi) the equation(s) of any horizontal asymptotes

E. What transformations to the graph of $y = f(x)$ result in the graph of $y = -\frac{1}{2}f(x + 2) - 1$?

Reflecting

F. For which parent functions are the domain, range, intervals of increase/decrease, and turning points affected when their graphs are transformed?

G. Describe the most efficient order that can be used to graph a transformed function when performing multiple transformations.

H. The most general equation of a transformed function is $y = af(k(x - d)) + c$, where a, k, c, and d are real numbers. Describe the transformations that would be performed on the parent function $y = f(x)$ in terms of the parameters a, k, c, and d.

APPLY the Math

EXAMPLE 1 Connecting transformations to the equation of a function

State the function that would result from vertically compressing $y = f(x)$ by a factor of $\frac{1}{2}$ and then translating the graph 5 units to the right.

Solution

$y = \frac{1}{2}f(x)$

This is the function that has a vertical compression by a factor of $\frac{1}{2}$.

$y = \frac{1}{2}f(x - 5)$

This is the function has also has a translation 5 units to the right.

EXAMPLE 2 Connecting transformations to the characteristics of a function

Use transformations to help you describe the characteristics of the transformed function $y = 3\sqrt{x} - 2$.

Solution

In the general function $y = af(k(x - d)) + c$, the parameters k and d affect the x-coordinates of each point on the parent function, and the parameters a and c affect the y-coordinates. Each point (x, y) on the parent function is mapped onto $\left(\frac{x}{k} + d, ay + c\right)$ on the transformed function.

The parameters k and a are related to stretches/compressions and reflections, while the parameters d and c are related to translations. Since division and multiplication must be performed before addition, all stretches/compression and reflections must be applied before any translations, due to the order of operations.

The equation $y = 3\sqrt{x} - 2$ indicates that two transformations have been applied to the parent function $y = \sqrt{x}$:

In this equation, $a = 3$ and $c = -2$.

1. a vertical stretch by a factor of 3
2. a vertical translation 2 units down

$(x, y) \rightarrow (x, 3y)$

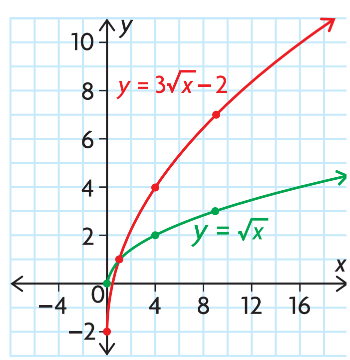
Parent Function $y = \sqrt{x}$	Stretched Function $y = 3\sqrt{x}$
$(0, 0)$	$(0, 3(0)) = (0, 0)$
$(1, 1)$	$(1, 3(1)) = (1, 3)$
$(4, 2)$	$(4, 3(2)) = (4, 6)$
$(9, 3)$	$(9, 3(3)) = (9, 9)$

Vertically stretching the graph by a factor of 3 occurs when all the y-coordinates on the graph of the parent function are multiplied by 3.

$(x, 3y) \rightarrow (x, 3y - 2)$

Stretched Function $y = 3\sqrt{x}$	Final Transformed Function $y = 3\sqrt{x} - 2$
$(0, 0)$	$(0, 0 - 2) = (0, -2)$
$(1, 3)$	$(1, 3 - 2) = (1, 1)$
$(4, 6)$	$(4, 6 - 2) = (4, 4)$
$(9, 9)$	$(9, 9 - 2) = (9, 7)$

Translating the graph 2 units down occurs when 2 is subtracted from all the y-coordinates on the graph of the stretched function.



Plot the key points of $y = \sqrt{x}$ and the new points of the transformed function.

Since the domain of both the parent function and transformed function is the same, the interval of increase is also the same: $[0, \infty)$. The difference occurs in the range. The y-values of the transformed function increase faster than the y-values of the parent function.

These two transformations act on the y values only; there is no change to the x values. The domain is unchanged; it is $\{x \in \mathbf{R} \mid x \geq 0\}$. The range changes from $\{y \in \mathbf{R} \mid y \geq 0\}$ to $\{y \in \mathbf{R} \mid y \geq -2\}$.

EXAMPLE 3 Reasoning about the characteristics of a transformed function

Graph the function $f(x) = \cos(x)$ and the transformed function $y = 2f(3x)$, where $0° \leq x \leq 360°$. State the impact of the transformations on the domain, range, intervals of increase/decrease, and turning points of the transformed function.

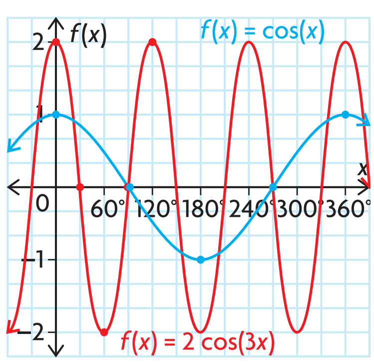
Solution

$$(x, y) \rightarrow \left(\frac{1}{3}x, 2y\right)$$

Parent Function $y = \cos(x)$	Final Transformed Function $y = 2\cos(3x)$
$(0°, 1)$	$\left(\frac{1}{3}(0°), 2(1)\right) = (0°, 2)$
$(90°, 0)$	$\left(\frac{1}{3}(90°), 2(0)\right) = (30°, 0)$
$(180°, -1)$	$\left(\frac{1}{3}(180°), 2(-1)\right) = (60°, -2)$
$(270°, 0)$	$\left(\frac{1}{3}(270°), 2(0)\right) = (90°, 0)$
$(360°, 1)$	$\left(\frac{1}{3}(360°), 2(1)\right) = (120°, 2)$

Apply a horizontal compression by a factor of $\frac{1}{3}$ and a vertical stretch by a factor of 2.

On the graph of $f(x) = \cos(x)$, multiply the x-coordinates by $\frac{1}{3}$ and the y-coordinates by 2.



Plot the key points of the parent function and the transformed points.

Within the specified domain,

- the transformed function decreases on the intervals $(0°, 60°)$, $(120°, 180°)$, and $(240°, 300°)$ and increases on the intervals $(60°, 120°)$, $(180°, 240°)$, and $(300°, 360°)$
- the transformed function has the following turning points: $(60°, -2)$, $(120°, 2)$, $(180°, -2)$, $(240°, 2)$, and $(300°, -2)$

The domain consists of all real numbers; this is not changed by the horizontal compression and translation.
Domain $= \{x \in \mathbf{R}\}$.
The vertical stretch has changed the range from $\{y \in \mathbf{R} \mid -1 \leq y \leq 1\}$ to $\{y \in \mathbf{R} \mid -2 \leq y \leq 2\}$.

EXAMPLE 4 Reasoning about the order of transformations

Describe the order in which you would apply the transformations defined by $y = -2f(3(x+1)) - 4$ to $f(x) = \sqrt{x}$. Then state the impact of the transformations on the domain, range, intervals of increase/decrease, and end behaviours of the transformed function.

Solution

$$(x, y) \rightarrow \left(\frac{1}{3}x, -2y\right)$$

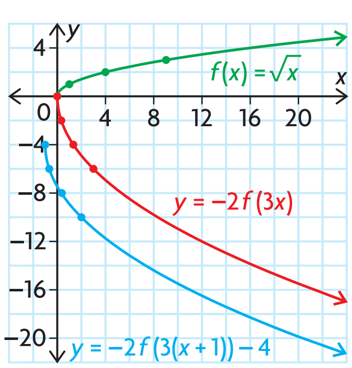
Parent Function $y = \sqrt{x}$	Stretched/Compressed Function $y = -2\sqrt{3x}$
$(0, 0)$	$\left(\frac{1}{3}(0), -2(0)\right) = (0, 0)$
$(1, 1)$	$\left(\frac{1}{3}(1), -2(1)\right) = \left(\frac{1}{3}, -2\right)$
$(4, 2)$	$\left(\frac{1}{3}(4), -2(2)\right) = \left(\frac{4}{3}, -4\right)$
$(9, 3)$	$\left(\frac{1}{3}(9), -2(3)\right) = (3, -6)$

Since multiplication must be done before addition, apply a horizontal compression by a factor of $\frac{1}{3}$, a vertical stretch by a factor of 2, and a reflection in the x-axis. To do this, multiply the x-coordinates of points on the parent function by $\frac{1}{3}$ and the y-coordinates by -2.

$$\left(\frac{1}{3}x, -2y\right) \rightarrow \left(\frac{1}{3}x - 1, -2y - 4\right)$$

Stretched/Compressed Function $y = -2\sqrt{3x}$	Final Transformed Function $y = -2\sqrt{3(x+1)} - 4$
$(0, 0)$	$(0 - 1, 0 - 4) = (-1, -4)$
$\left(\frac{1}{3}, -2\right)$	$\left(\frac{1}{3} - 1, -2 - 4\right) = \left(-\frac{2}{3}, -6\right)$
$\left(\frac{4}{3}, -4\right)$	$\left(\frac{4}{3} - 1, -4 - 4\right) = \left(\frac{1}{3}, -8\right)$
$(3, -6)$	$(3 - 1, -6 - 4) = (2, -10)$

Apply all translations next. Translate the graph of $f(x) = -2f(3x)$ 1 unit to the left and 4 units down. To do this, subtract 1 from the x-coordinates and 4 from the y-coordinates of points on the previous function.



The transformed function is now a decreasing function on the interval $[-1, \infty)$.
The transformed function has the following end behaviours:

As $x \rightarrow -1$, $y \rightarrow -4$ and as $x \rightarrow \infty$, $y \rightarrow -\infty$.

Plot the points of the final transformed function. The horizontal translation changed the domain from $\{x \in \mathbf{R} \mid x \geq 0\}$ to $\{x \in \mathbf{R} \mid x \geq -1\}$.

The reflection in the x-axis and the vertical translation changed the range from $\{y \in \mathbf{R} \mid y \geq 0\}$ to $\{y \in \mathbf{R} \mid y \leq -4\}$.

In Summary

Key Ideas

- Transformations on a function $y = af(k(x - d)) + c$ must be performed in a particular order: horizontal and vertical stretches/compressions (including any reflections) must be performed before translations. All points on the graph of the parent function $y = f(x)$ are changed as follows: $(x, y) \rightarrow \left(\frac{x}{k} + d, ay + c\right)$
- When using transformations to graph, you can apply a and k together, and then c and d together, to get the desired graph in the fewest number of steps.

Need to Know

- The value of a determines whether there is a vertical stretch or compression, or a reflection in the x-axis:
  - When $|a| > 1$, the graph of $y = f(x)$ is stretched vertically by the factor $|a|$.
  - When $0 < |a| < 1$, the graph is compressed vertically by the factor $|a|$.
  - When $a < 0$, the graph is also reflected in the x-axis.
- The value of k determines whether there is a horizontal stretch or compression, or a reflection in the y-axis:
  - When $|k| > 1$, the graph is compressed horizontally by the factor $\frac{1}{|k|}$.
  - When $0 < |k| < 1$, the graph is stretched horizontally by the factor $\frac{1}{|k|}$.
  - When $k < 0$, the graph is also reflected in the y-axis.
- The value of d determines whether there is a horizontal translation:
  - For $d > 0$, the graph is translated to the right.
  - For $d < 0$, the graph is translated to the left.
- The value of c determines whether there is a vertical translation:
  - For $c > 0$, the graph is translated up.
  - For $c < 0$, the graph is translated down.

CHECK Your Understanding

1. State the transformations defined by each equation in the order they would be applied to $y = f(x)$.

a) $y = f(x) - 1$

b) $y = f(2(x - 1))$

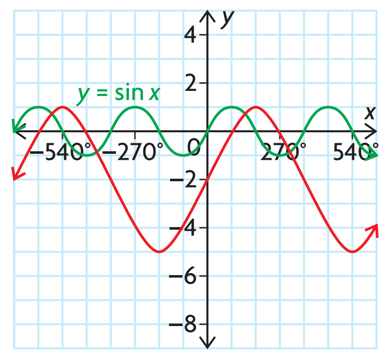
c) $y = -f(x - 3) + 2$

d) $y = -2f(4x)$

e) $y = -f(-(x + 2)) - 3$

f) $y = \frac{1}{2}f\left(\frac{1}{4}(x - 5)\right) + 6$

2. Identify the appropriate values for a, k, c, and d in $y = af(k(x - d)) + c$ to describe each set of transformations below.
   a) horizontal stretch by a factor of 2, vertical translation 3 units up, reflection in the x-axis
   b)



3. The point $(2, 3)$ is on the graph of $y = f(x)$. Determine the corresponding coordinates of this point on the graph of $y = -2(f(2(x + 5)) - 4$.

PRACTISING

4. The ordered pairs $(2, 3)$, $(4, 7)$, $(-2, 5)$, and $(-4, 6)$ belong to a function f. List the ordered pairs that belong to each of the following:
   a) $y = 2f(x)$
   b) $y = f(x - 3)$
   c) $y = f(x) + 2$
   d) $y = f(x + 1) - 3$
   e) $y = f(-x)$
   f) $y = f(2x) - 1$

5. K For each of the following equations, state the parent function and the transformation that was applied. Graph the transformed function.
   a) $y = (x + 1)^2$
   b) $y = 2|x|$
   c) $y = \sin(3x) + 1$
   d) $y = \dfrac{1}{x} + 3$
   e) $y = 2^{0.5x}$
   f) $y = \sqrt{2(x - 6)}$

6. State the domain and range of each function in question 5.

7. a) Graph the parent function $y = 2^x$ and the transformed function defined by $y = -2f(3(x - 1)) + 4$.
   b) State the impact of the transformations on the domain and range, intervals of increase/decrease, and end behaviours.
   c) State the equation of the transformed function.

8. The graph of $y = \sqrt{x}$ is stretched vertically by a factor of 3, reflected in the x-axis, and shifted 5 units to the right. Determine the equation that results from these transformations, and graph it.

9. The point $(1, 8)$ is on the graph of $y = f(x)$. Find the corresponding coordinates of this point on each of the following graphs.

a) $y = 3f(x - 2)$
b) $y = f(2(x + 1)) - 4$
c) $y = -2f(-x) - 7$
d) $y = -f(4(x + 1))$
e) $y = -f(-x)$
f) $y = 0.5f(0.5(x + 3)) + 3$

10. **A** Given $f(x) = \sqrt{x}$, find the domain and range for each of the following:

a) $g(x) = f(x - 2)$
b) $h(x) = 2f(x - 1) + 4$
c) $k(x) = f(-x) + 1$
d) $j(x) = 3f(2(x - 5)) - 3$

11. Greg thinks that the graphs of $y = 5x^2 - 3$ and $y = 5(x^2 - 3)$ are the same. Explain why he is incorrect.

12. Given $f(x) = x^3 - 3x^2$, $g(x) = f(x - 1)$, and $h(x) = -f(x)$, graph each function and compare $g(x)$ and $h(x)$ with $f(x)$.

13. **T** Consider the parent function $y = x^2$.

a) Describe the transformation that produced the equation $y = 4x^2$.
b) Describe the transformation that produced the equation $y = (2x)^2$.
c) Show algebraically that the two transformations produce the same equation and graph.

14. **C** Use a flow chart to show the sequence and types of transformations required to transform the graph of $y = f(x)$ into the graph of $y = af(k(x - d)) + c$.

Extending

15. The point $(3, 6)$ is on the graph of $y = 2f(x + 1) - 4$. Find the original point on the graph of $y = f(x)$.

16. **a)** Describe the transformations that produce $y = f(3(x + 2))$.

b) The graph of $y = f(3x + 6)$ is produced by shifting 6 units to the left and then compressing the graph by a factor of $\frac{1}{3}$. Why does this produce the same result as the transformations you described in part a)?

c) Using $f(x) = x^2$ as the parent function, graph the transformations described in parts a) and b) to show that they result in the same transformed function.

1.5 Inverse Relations

YOU WILL NEED

- graph paper
- graphing calculator

GOAL

Determine the equation of an inverse relation and the conditions for an inverse relation to be a function.

LEARN ABOUT the Math

The owners of a candy company are creating a spherical container to hold their small chocolates. They are trying to decide what size to make the sphere and how much volume the sphere will hold, based on its radius.

The volume of a sphere is given by the relationship $V = \frac{4}{3}\pi r^3$.

? How can you use this relationship to find the radius of any sphere for a given volume?

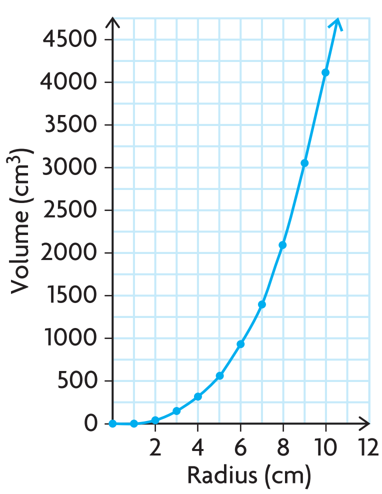
EXAMPLE 1 Representing the inverse using a table of values and a graph

Use a table of values and a graphical model to represent the relationship between the radius of a sphere and any given volume.

Solution

$$V = \frac{4}{3}\pi r^3$$

Radius (cm)	Volume (cm^3)
0.0	0.0
1.0	4.2
2.0	33.5
3.0	113.1
4.0	268.1
5.0	523.6
6.0	904.8
7.0	1436.8
8.0	2144.7
9.0	3053.6
10.0	4188.8

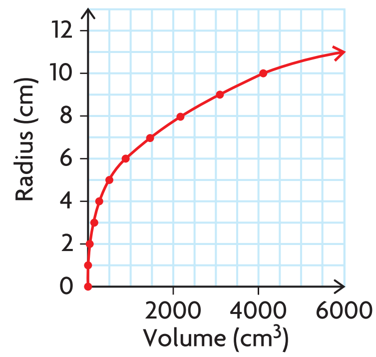


Radius is the independent variable, and volume is the dependent variable.

Create a table of values, and calculate the volume for a specific radius.

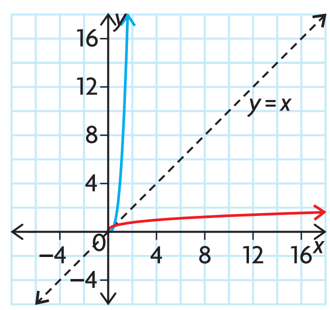
Draw a scatter plot of volume in terms of radius. Draw a smooth curve through the points since the function is continuous.

Volume (cm^3)	Radius (cm)
0.0	0
4.2	1
33.5	2
113.1	3
268.1	4
523.6	5
904.8	6
1436.8	7
2144.7	8
3053.6	9
4188.8	10



To graph radius in terms of volume, switch the variables in the table, making radius the dependent variable and volume the independent variable.

The red curve shows volume as the independent variable and radius as the dependent variable.



If we ignore units and plot both relations on the same graph, the red curve is a reflection of the blue curve in the line $y = x$. This is reasonable, given that the x-values and y-values were switched on the graph. The red curve is the inverse relation, and it is also a function.

The inverse was found by switching the independent and dependent variables in the table of values. The independent and dependent variables can also be switched in the equation of the relation to determine the equation of the inverse relation.

EXAMPLE 2 Representing the inverse using an equation

Recall that the volume of a sphere is given by the relationship $V = \frac{4}{3}\pi r^3$.

Determine the equation of the inverse.

Solution

$$V = \frac{4}{3}\pi r^3$$

To express V in terms of r, rearrange the formula using inverse operations.

$$3 \times V = 3 \times \left(\frac{4}{3}\pi r^3\right)$$

Multiply both sides by 3 to eliminate the fraction.

$$3V = 4\pi r^3$$

$$\frac{3V}{4\pi} = \frac{4\pi r^3}{4\pi}$$

$$\sqrt[3]{\frac{3V}{4\pi}} = \sqrt[3]{r^3}$$

Divide both sides by 4π (the coefficient of r^3) to isolate r^3. Take the cube root of both sides to isolate r.

$$\sqrt[3]{\frac{3V}{4\pi}} = r$$

The radius is now expressed as a function of volume and can be determined for different values of V.

Reflecting

A. Compare the domain and range of this function and its inverse.

B. Will an **inverse of a function** always be a function? Explain.

C. Why is it reasonable to switch the V and the r in Example 2 to determine the inverse relation?

APPLY the Math

EXAMPLE 3 Using an algebraic strategy to determine the inverse relation

Given $f(x) = x^2$.

a) Find the inverse relation.
b) Compare the domain and range of the function and its inverse.
c) Determine if the inverse relation is also a function.

Solution

a)

$$y = x^2$$

Rewrite the function using x and y.

$$x = y^2$$

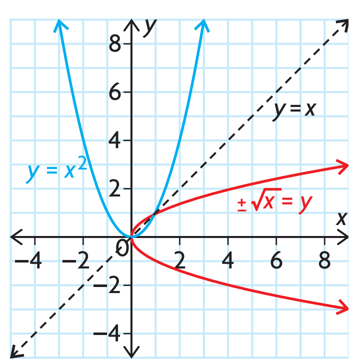
Interchange x and y in the relation.

$$\pm\sqrt{x} = \sqrt{y^2}$$

Solve for y by taking the square root of both sides.

$$\pm\sqrt{x} = y$$

b)



The graph of the inverse relation is a reflection of the original relation in the line $y = x$.

Only non-negative values of x work in the square root function. The square root of a negative number is undefined. Since $\pm$ in the inverse indicates that the output, y, will include both positive and negative values, the range will include all the real numbers.

The domain of $y = x^2$ is $\{x \in \mathbf{R}\}$. The range is $\{x \in \mathbf{R} \mid y \geq 0\}$.

The domain of the inverse relation is $\{x \in \mathbf{R} \mid x \geq 0\}$. The range is $\{y \in \mathbf{R}\}$.

c) The inverse relation is not a function, but it can be split in the middle into the two functions, $y = \sqrt{x}$ and $y = -\sqrt{x}$.

Based on the equation of the inverse relation, each input of x will have two outputs for y, one positive and one negative. The only exception is $x = 0$.

Communication *Tip*

The domain of the square root function is $\{x \in \mathbf{R} \mid x \geq 0\}$; we say the values of x are non-negative. The range of the exponential function $y = 2^x$ is $\{y \in \mathbf{R} \mid y > 0\}$; we say the values of y are positive. The distinction is because zero is neither negative nor positive.

The inverse relation is useful to solve problems, particularly when you are given a value of the dependent variable and need to determine the value of the corresponding independent variable.

EXAMPLE 4 Selecting a strategy that involves the inverse relation to solve a problem

Archaeologists use models for the relationship between height and footprint length to determine the height of a person based on the lengths of the bones they discover. The relationship between height, $h(x)$, in centimetres and footprint length, x, in centimetres is given by $h(x) = 1.1x + 143.6$. Use this relationship to predict the footprint length for a person who is 170 cm tall.

Solution

$$h(x) = 1.1x + 143.6$$

Let $y = h(x)$.

$$y = 1.1x + 143.6$$

To predict the footprint length, rewrite the relationship with footprint length as the dependent variable and $h(x)$ as the independent variable.

$$x = 1.1y + 143.6$$

Interchange x and y.

$$x - 143.6 = 1.1y$$

$$\frac{x - 143.6}{1.1} = y = h^{-1}(x)$$

Solve for y.

$$h^{-1}(170) = \frac{170 - 143.6}{1.1}$$
$$= 24 \text{ cm}$$

Evaluate $h^{-1}(170)$.

A person who is 170 cm tall may have a footprint length of 24 cm.

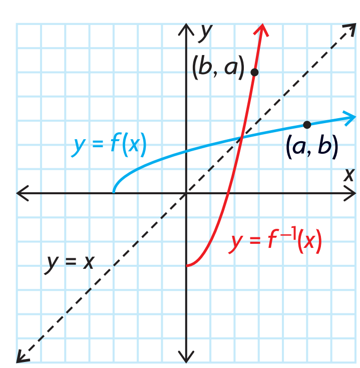
Communication *Tip*

When an inverse relation is also a function, the notation $f^{-1}(x)$ can be used to define the inverse function.

In Summary

Key Ideas

- The inverse function of $f(x)$ is denoted by $f^{-1}(x)$. Function notation can only be used when the inverse is a function.
- The graph of the inverse function is a reflection in the line $y = x$.



(continued)

Need to Know

- Not all inverse relations are functions. The domain and/or range of the original function may need to be restricted to ensure that the inverse of a function is also a function.
- To find the inverse algebraically, write the function equation using y instead of $f(x)$. Interchange x and y. Solve for y.
- If (a, b) represents a point on the graph of $f(x)$, then (b, a) represents a point on the graph of the corresponding f^{-1}.
- Given a table of values or a graph of a function, the independent and dependent variables can be interchanged to get a table of values or a graph of the inverse relation.
- The domain of a function is the range of its inverse. The range of a function is the domain of its inverse.

CHECK Your Understanding

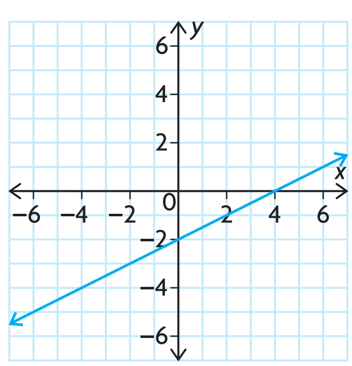
1. Each of the following ordered pairs is a point on a function. State the corresponding point on the inverse relation.

a) $(2, 5)$ **c)** $(4, -8)$ **e)** $g(-3) = 0$
b) $(-5, -6)$ **d)** $f(1) = 2$ **f)** $h(0) = 7$

2. Given the domain and range of a function, state the domain and range of the inverse relation.

a) $D = \{x \in \mathbf{R}\}, R = \{y \in \mathbf{R}\}$
b) $D = \{x \in \mathbf{R} \mid x \geq 2\}, R = \{y \in \mathbf{R}\}$
c) $D = \{x \in \mathbf{R} \mid x \geq -5\}, R = \{y \in \mathbf{R} \mid y < 2\}$
d) $D = \{x \in \mathbf{R} \mid x < -2\}, R = \{y \in \mathbf{R} \mid -5 < y < 10\}$

3. Match the inverse relations to their corresponding functions.

A


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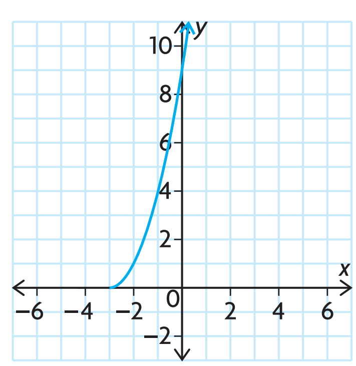

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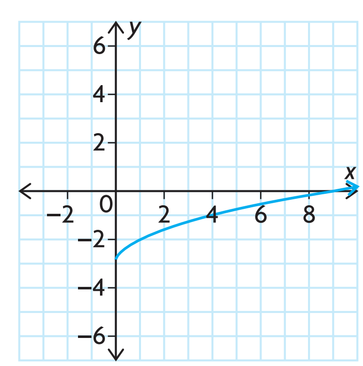

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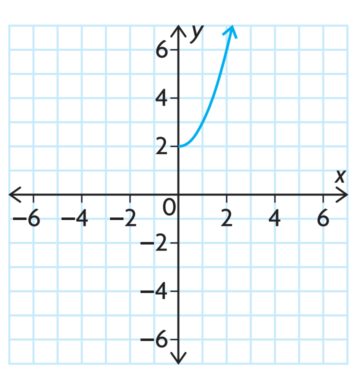

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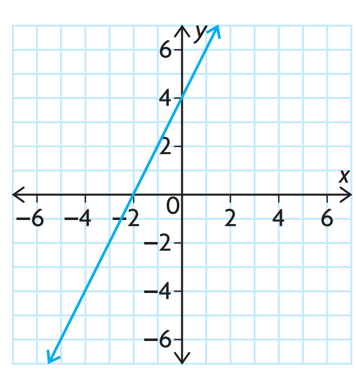

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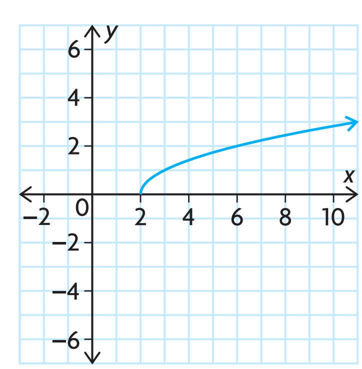

PRACTISING

4. Consider the function $f(x) = 2x^3 + 1$.

K

a) Find the ordered pair $(4, f(4))$ on the function.

b) Find the ordered pair on the inverse relation that corresponds to the ordered pair from part a).

c) Find the domain and range of f.

d) Find the domain and the range of the inverse relation of f.

e) Is the inverse relation a function? Explain.

5. Repeat question 4 for the function $g(x) = x^4 - 8$.

6. Graph each function and its inverse relation on the same set of axes. Determine whether the inverse relation is a function.

a) $f(x) = x^2 + 1$

b) $g(x) = \sin x$, where $-360° \le x \le 360°$

c) $h(x) = -x$

d) $m(x) = |x| + 1$

7. a) The equation $F = \frac{9}{5}C + 32$ can be used to convert a known Celsius temperature, C, to the equivalent Fahrenheit temperature, F. Find the inverse of this relation, and describe what it can be used for.

A

b) Use the equation given in part a) to convert 20 °C to its equivalent Fahrenheit temperature. Use the inverse relation to convert this Fahrenheit temperature back to its equivalent Celsius temperature.

8. a) The formula $A = \pi r^2$ is convenient for calculating the area of a circle when the radius is known. Find the inverse of the relation, and describe what it can be used for.

b) Use the equation given in part a) to calculate the area of a circle with a radius of 5 cm. (Express the area as an exact value in terms of π.) Use the inverse relation to calculate the radius of the circle with the area you calculated.

9. If $f(x) = kx^3 - 1$ and $f^{-1}(15) = 2$, find k.

T

10. Given the function $h(x) = 2x + 7$, find

a) $h(3)$

b) $h(9)$

c) $\dfrac{h(9) - h(3)}{9 - 3}$

d) $h^{-1}(3)$

e) $h^{-1}(9)$

f) $\dfrac{h^{-1}(9) - h^{-1}(3)}{9 - 3}$

11. Suppose that the variable a represents a particular student and $f(a)$ represents the student's overall average in all their subjects. Is the inverse relation of f a function? Explain.

12. Determine the inverse of each function.

a) $f(x) = 3x + 4$ **c)** $g(x) = x^3 - 1$
b) $h(x) = -x$ **d)** $m(x) = -2(x + 5)$

13. A function g is defined by $g(x) = 4(x - 3)^2 + 1$.

a) Determine an equation for the inverse of $g(x)$.
b) Solve for y in the equation for the inverse of $g(x)$.
c) Graph $g(x)$ and its inverse using graphing technology.
d) At what points do the graphs of $g(x)$ and its inverse intersect?
e) State **restrictions** on the domain or range of g so that its inverse is a function.
f) Suppose that the domain of $g(x)$ is $\{x \in \mathbf{R} \mid 2 \le x \le 5\}$. Is the inverse a function? Justify your answer.

14. A student writes, "The inverse of $y = -\sqrt{x + 2}$ is $y = x^2 - 2$." Explain why this statement is not true.

15. Do you have to restrict either the domain or the range of the function $y = \sqrt{x + 2}$ to make its inverse a function? Explain.

16. **C** John and Katie are discussing inverse relationships. John says, "A function is a rule, and the inverse is the rule performed in reverse order with opposite operations. For example, suppose that you cube a number, divide by 4, and add 2. The inverse is found by subtracting 2, multiplying by 4, and taking the cube root." Is John correct? Justify your answer algebraically, numerically, and graphically.

Extending

17. $f(x) = x$ is an interesting function because it is its own inverse. Can you find three more functions that have the same property? Can you convince yourself that there are an infinite number of functions that satisfy this property?

18. The inverse relation of a function is also a function if the original function passes the horizontal line test (in other words, if any horizontal line hits the function in at most one location). Explain why this is true.